3.3.4 Conjugating Surfaces of Velocity Discontinuities

In the following study, several upper-bound solutions for the case of constant shear friction factor are derived simultaneously for the power involved in drawing and in extrusion. Some general rules pertaining to the velocity fields leading to these solutions are presented here in Section 3.4. As noted in Section 3.3, at the vicinity of the die, the wire undergoes plastic deformation. Further away, at the entrance and exit of the die, the incoming workpiece and the product, respectively, move as two rigid bodies in a direction parallel to the axis of symmetry. The incoming rod moves at the velocity \( v_o \) and the product at the velocity \( v_f \). This is a steady state process; therefore, when a sound product is produced, the law of incompressibility dictates that:

\[
\frac{v_f}{v_o} = \left(\frac{R_o}{R_f}\right)^2
\]

Eq. (a)

When sound flow is maintained and a sound product is manufactured, it can be noted that any concentric cylinder of the radius \( R_1 \) in the raw material will end up as a concentric cylinder in the product. Volume constancy dictates that the ratio of the radius of the cylinder on the product \( R_2 \) to that of the cylinder at the entrance \( R_1 \) will equal the ratio of the radius of the product to that of the entering rod:

\[
\frac{R_2}{R_1} = \frac{R_f}{R_o}
\]

Eq. (b)

In the present study, a number of flow patterns within the deformation regions are explored. The relationships of the boundaries between the deformation region and its rigid body neighbors at the entrance and exit of the die to the characteristics of several flow patterns are investigated. The effectiveness of the several modes of deformation as models resembling the actual process is studied.

In the application of upper-bound solutions to the metalforming processes that employ flow-through conical converging dies, several velocity fields can be explored. Some of the literature that provides established solutions will be presented. In this introduction, classification of a wide range of kinematically admissible velocity fields is proposed. Figure <11> presents basic simple fields.

Some assembly arrangements of the simple fields that cover the entire region by a combination of several of the above mentioned simple fields can be described. As the assembly becomes more complex, the treatment gets closer and closer to the treatment of metalforming by numerical methods, specifically, the finite element method and the "Upper Bound Elemental Technique" (UBET).

In the present study, the boundaries between the deformation region (Zone II where plastic flow occurs) and its rigid neighbors at the entrance and exit of the die (Zones I and III,
respectively) are surfaces of velocity discontinuity, \( \Gamma_2 \) and \( \Gamma_1 \), respectively. A general flow pattern is described by the flow line in Fig. <11b>. Here, point A, at a radial distance \( R_1 \) from the axis of symmetry, moves toward the die at a velocity \( v_o \) until it reaches the surface of velocity discontinuity \( \Gamma_2 \). Undergoing a drastic change in direction and magnitude, the point crosses the surface \( \Gamma_2 \), enters the deformation region Zone II and moves in a generally converging flow towards the exit at an accelerated speed. The path of flow through the deformation region, Zone II, is described here as a winding general hypothetical path. On reaching the surface of velocity discontinuity \( \Gamma_1 \) at the exit, another change in direction and magnitude of the velocity vector occurs, and the point, on crossing \( \Gamma_1 \), proceeds to move parallel to the axis of symmetry at the constant rigid body speed of \( v_f \).

The radial distances of the point when it enters and exits Zone II are \( R_1 \) and \( R_2 \). Their ratio is dictated by Eq. (b) to be \( R_2/R_1 = R_f/R_o \).

The points along the path of flow where drastic changes in the velocity vector occur, that is, where point A crosses the surfaces of velocity discontinuity \( \Gamma_2 \) and \( \Gamma_1 \), are noted in Fig. <11b>. These two points are connected by a straight line which is extended to its intersection with the axis of symmetry of the workpiece and die. The intersection point is denoted by \( o \), and the radial
distance of the two points of $\Gamma_2$ and $\Gamma_1$, respectively, from o are denoted as the distances $r_0^*$ and $r_f^*$. From geometrical similarity and by Eq. (b), it is observed

$$\frac{r_0^*}{r_f^*} = \frac{R_1}{R_2} = \frac{R_o}{R_f}$$

Eq. (c)

The two surfaces, $\Gamma_1$ and $\Gamma_2$, are "conjugating surfaces" related to each other by Equation (c). This observation is most useful in further derivations, especially during the volume integration performed to determine the internal power of deformation in the deformation region.

In all of the following examples where "conjugating surfaces" (radial and parallel flow) are explored, the volume integral is performed first along a flow line from one surface of velocity discontinuity to the other, and then in the direction normal to the flow line and to the circumferential direction. The integration along a flow line results in the expression of $\ln(R_o/R_f)$, so that in all of the following solutions the internal power of deformation can be presented by

$$W_i = F(\alpha) \ln\left[ \frac{R_o}{R_f} \right]$$

Eq. (d)

where the function $F(\alpha)$ varies according to the assumed surfaces (and velocities).

The basic flow modes that are described in this manuscript maintain only straight-line flow in the deformation region. Thus, for solid rod, but not for tube, the conjugation rule provided by Eq. (c) takes on a simple significance and is described with each individual field.

The velocity fields described in Fig. <11a> are a radially converging flow directed into the apex of the die (o). The two surfaces of velocity discontinuity, $\Gamma_1$ and $\Gamma_2$, are "parallel" so that the ratio of the radial distance ($r_0^*$) of a point entering the deformation region to its radial distance ($r_f^*$) exiting the deformation region is equal to the ratio $R_o/R_f$. The angular position remains constant for each point while it is passing through the plastic region. The angle $\beta$ that the tangential line to the curved surface of discontinuity makes with the axes of symmetry may be chosen as an arbitrary function of the angular position. The velocity of the point entering the deformation region with coordinates $r_0^*$ and $\theta$ is shown in Eq. (e).

$$\dot{U}_r |_{r_2} = v_0 \frac{\sin(\beta)}{\sin(\beta + \theta)}$$

Eq. (e)

In the deformation region, radially converging flow coupled with the law of incompressibility leads to

$$\dot{U}_r = \dot{U}_r |_{r_2} \left[ \frac{r_0^*}{r} \right]^2 = v_0 \frac{r_0^*}{r} \sin(\beta) = v_f \frac{r_f^*}{r} \sin(\beta)$$

Eq. (f)
Note that for the spherical field where \( r_o^* \) and \( r_f^* \) are constants, Eq. (f) reduces to

\[
\dot{U}_r = v_f \left[ \frac{r_f^*}{r} \right] \cos(\theta)
\]

Eq. (g)

Along surface \( \Gamma_1 \) and \( \Gamma_2 \), the velocity discontinuities are

\[
\Delta v |_{\Gamma_2} = v_o \cos(\beta) - \dot{U}_r \cos(\beta + \theta) = v_f \left[ \cos(\beta) + \sin(\beta) \cot(\beta + \theta) \right]
\]

\[
\Delta v |_{\Gamma_1} = v_f \left[ \cos(\beta) + \sin(\beta) \cot(\beta + \theta) \right]
\]

Eq. (h)

For the spherical velocity field

\[
\Delta v |_{\Gamma_1} = v_o \cos(\beta) = v_o \sin(\theta)
\]

\[
\Delta v |_{\Gamma_2} = v_f \sin(\theta)
\]

Eq. (i)

Because \( (r_f^*) \) in Eq. (f) is yet an arbitrary function of \( \theta \), \( (r_f^* = r_f^*(\theta)) \), the determination of the strain rate components by applying Eq. (8.1) of Ref. [12] to Eq. (f) is not performed here. When any specific functional relation of \( r_f^* \) to \( \theta \) is determined, the strain rate components are derivable.

The basic modes of converging flow described in Fig. <11> can be classified as conically converging radial flow and parallel flow.

**Conically Converging Radial Flow:** In the general conically converging radial flow, above, any point in the deformation Zone II moves towards the apex of the die, \( o \), along a straight line which forms the angle \( \theta \) with the axis of symmetry of the workpiece and die. A generally winding shape of one surface of velocity discontinuity, surface \( \Gamma_1 \), is admissible, dictating the shape of the other conjugate surface through Eq. (c). These surfaces are called "conjugating similar surfaces." The shapes of the surfaces of velocity discontinuity dictate the dependence of \( r_o^* \) and \( r_f^* \) on the angle \( \theta \).

Two specific shapes of the surfaces of velocity discontinuity are considered. First, the spherical surfaces of velocity discontinuities, associated with the so-called "spherical velocity field" are denoted hereafter by the symbol (o) (see Refs. [2], [4], [12] and [13]). The surfaces \( \Gamma_1 \) and \( \Gamma_2 \) are two concentric spherical surfaces with their center at the apex (o) of the die. The radial distances \( r_o^* \) and \( r_f^* \) are independent of \( \theta \). The spherical velocity field is a rigid single field, not a family of fields, and has no pseudo-independent parameter by which the computed power can be minimized.

The second specific field in the category of conically converging radial flows is the "trapezoidal" field (\( \square \)) with parallel surfaces \( \Gamma_1 \) and \( \Gamma_2 \) inclined to the axis of symmetry at an
arbitrary angle $\beta$. The angle $\beta$ becomes a pseudo-independent parameter subject to optimization. The treatment of the trapezoidal field for drawing and extrusion in plane strain is presented in Refs. [70] and [71].

**Parallel Flow:** In the parallel flow, any point in the deformation region in Fig. <11c>, is moving along a straight line parallel to the surface $\Gamma_3$ which may be the surface of the die. In a more complex assembly of units, the surface $\Gamma_3$ may be an interface between the two fields.

The surfaces $\Gamma_1$ and $\Gamma_2$ can be curved as long as the vertices (edges) are at the entrance, the exit and on the axis of symmetry, point P, respectively, and the ratio of $r_o^*/r_f^* = R_o/R_f$ (Eq. (c)) is obtained for each point passing through the deformation region. These surfaces ($\Gamma_1$ and $\Gamma_2$) are called "conjugate intersecting surfaces." The radial distances $r_o^*$ and $r_f^*$ become a general function of the position of the moving apex o. In this manuscript, the surfaces $\Gamma_1$ and $\Gamma_2$ are handled only as straight lines and the position of the vertex P is treated as a pseudo-independent parameter, subject to optimization. The field is called the "triangular" field denoted by $\Delta$.

**The Toroidal Flow:** In the toroidal flow, (Fig. <11d>), the straight-line flow is directed towards the apex o, which is a circle of a radius e lying on the extension of the conical surface $\Gamma_3$. One surface ($\Gamma_2$ or $\Gamma_1$) may be a toroidal surface with the center at o, but then the other surface ($\Gamma_1$ or $\Gamma_2$, respectively) becomes a complex function of e and $\theta$. When this field applies to the analysis of tubes, it leads to singularity problems for finite non-zero values of e when $\theta$ approaches zero. The complexity of the solution excludes the treatment of the toroidal field from this text.

The spherical velocity field is handled next because of its relative simplicity, of the simplicity of its derivation and of the treatment of its results. It is presently widely accepted, it is most useful and it is one of the earliest fields proposed with the introduction of the upper-bound approach. In its present form, it is proposed as one alternative for the conditions prevailing in the vicinity of the transition from converging flow ahead of the die and a flow with a bulge. The other solutions for this range, namely the triangular and the trapezoidal velocity fields, are treated in detail in Section {9.3} and are compared there with the solution by the spherical velocity field.